On Testing for the Equality of Autocovariance Between Time Series

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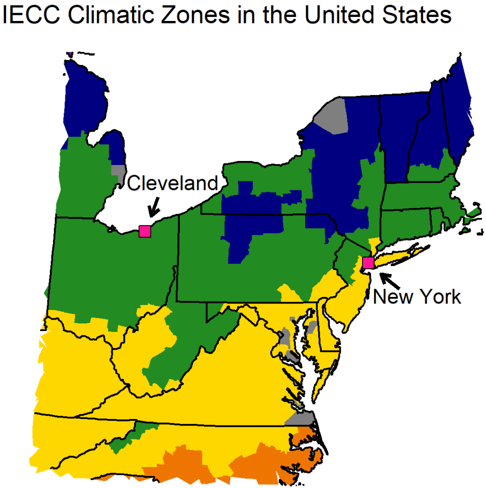
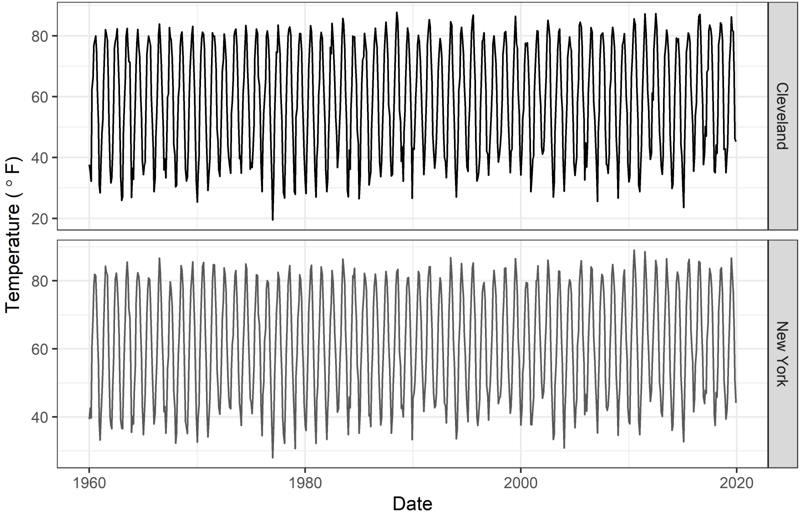
Motivating Example

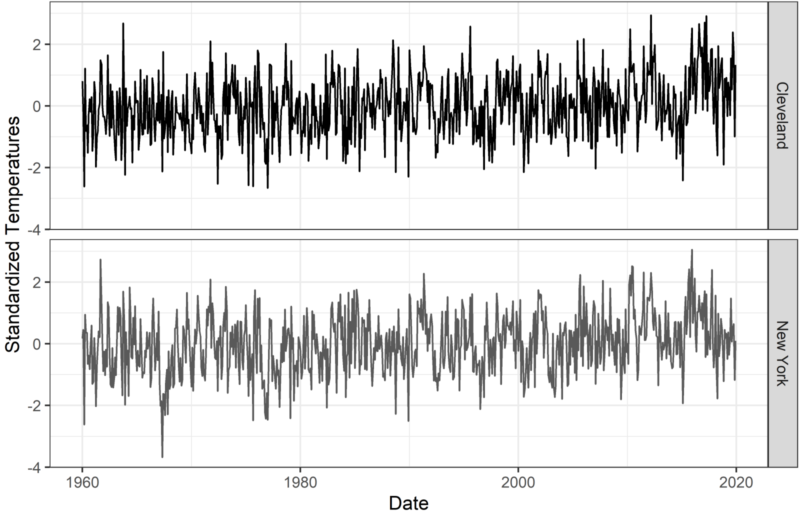


US Dep of Energy: International Energy Conservation Code

Mean monthly high temperatures



Stationary series (standardized within month)



- Standardize by monthly means and standard deviations

Estimation of the ACVF

Definition (Sample Autocovariance Function)

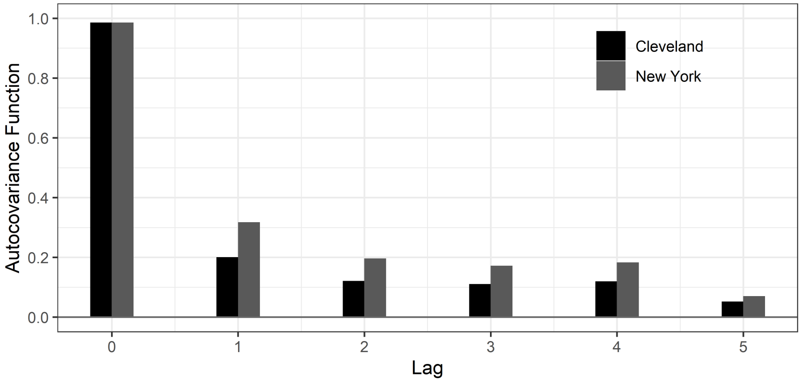
We compare the **sample autocovariance function** of a stationary time series is given by the

$$\hat{\gamma}_X(k) = \frac{1}{n}\sum_{t=k+1}^{n}(X_t - \bar{X})(X_{t-k} - \bar{X})$$

for $k = 0, 1, 2, \ldots n-1$

- This estimator has very nice asymptotic properties that will be useful later

The Sample ACVF of the NY and CLE series



- How do we test for a difference?

Caveats

We present the results today in a **Univariate** framework but all the results presented herein can be extended to the **Multivariate** setting with relative ease (careful bookkeeping). https://tjfisher19.github.io/research.html We also study related

problems which we will recap later.

Details can be found in Cirkovic and Fisher [2021]

Compare the autocovariance functions

Lund et al. [2009] developed a test for equality of autocovariance functions up to a pre-specified lag L for two stationary, independent time series and showed (through simulation) that it performed better than existing frequency domain tests.

The test is given by

$$H_0 : \begin{bmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \vdots \\ \gamma_X(L) \end{bmatrix} = \begin{bmatrix} \gamma_Y(0) \\ \gamma_Y(1) \\ \vdots \\ \gamma_Y(L) \end{bmatrix} \qquad H_A : \begin{bmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \vdots \\ \gamma_X(L) \end{bmatrix} \neq \begin{bmatrix} \gamma_Y(0) \\ \gamma_Y(1) \\ \vdots \\ \gamma_Y(L) \end{bmatrix}$$

Construction of the Lund et al. [2009] test

The test is based on Bartlett's result

Theorem (Bartlett's Result)

If X_t is the zero mean stationary process

$$X_t = \sum_{k=-\infty}^{\infty} \psi_k Z_{t-k} \quad Z_t \sim IID(0, \sigma^2)$$

where $\sum_{k=-\infty}^{\infty} |\psi_k| < \infty$ *and* $E[Z_t^4] < \infty$ *then given some fixed L*

$$\sqrt{n} \left(\begin{bmatrix} \hat{\gamma}_X(0) \\ \hat{\gamma}_X(1) \\ \vdots \\ \hat{\gamma}_X(L) \end{bmatrix} - \begin{bmatrix} \gamma_X(0) \\ \gamma_X(1) \\ \vdots \\ \gamma_X(L) \end{bmatrix} \right) \xrightarrow{d} MVN\,(0, \mathrm{W})$$

Construction of the Lund et al. [2009] test

Theorem (Bartlett's Formula)

If Z_t is Gaussian, we can compute the entries of W *by*

$$W_{i,j} = \sum_{k=-\infty}^{\infty} \gamma(k)\gamma(k-i+j) + \gamma(k+j)\gamma(k-i)$$

Brockwell et al. [1991]

Construction of the Lund et al. [2009] test

Then, under the null hypothesis of equality of autocovariances

$$\sqrt{n}\hat{\Delta} = \sqrt{n}\begin{bmatrix}\hat{\gamma}_X(0) - \hat{\gamma}_Y(0)\\ \hat{\gamma}_X(1) - \hat{\gamma}_Y(1)\\ \vdots \\ \hat{\gamma}_X(L) - \hat{\gamma}_Y(L)\end{bmatrix} \xrightarrow{d} MVN(0, 2\mathrm{W})$$

Hence, under the null

$$U_L = \frac{n}{2}\hat{\Delta}`\mathrm{W}^{-1}\hat{\Delta} \xrightarrow{d} \chi^2_{L+1}$$

as $n \to \infty$.

Construction of the Lund et al. [2009] test

This gives rise to a hypothesis test of equality of covariances that rejects the null hypothesis when

$$\hat{U}_L = \frac{n}{2}\hat{\Delta}`\hat{W}^{-1}\hat{\Delta}$$

exceeds the $(1-\alpha)100$th quantile of a χ^2_{L+1} distribution.

W is estimated by employing the null hypothesis estimate $\hat{\gamma}(k) = \frac{1}{2}(\hat{\gamma}_X(k) + \hat{\gamma}_Y(k))$ and truncating the covariance estimate

$$\hat{W}_{i,j} = \sum_{k=-\lfloor n^{1/3}\rfloor}^{\lfloor n^{1/3}\rfloor} \hat{\gamma}(k)\hat{\gamma}(k-i+j) + \hat{\gamma}(k+j)\hat{\gamma}(k-i)$$

Where the truncation rule follows the conditions of Theorem A.1 in Berkes et al.

Application of the Test to the NYC, CLE data

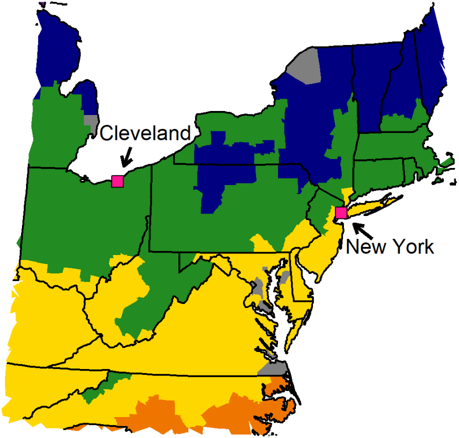
Test	L	Statistic	df	p
U_L	5	6.340	6	0.386

Table: Test of Lund et al. [2009] for Equality of Autocovariances up to Lag L

- This suggest the autocovariance functions are equivalent
- This does not match our expectations (different climate zones)
- Are we violating assumptions?

The Cities

IECC Climatic Zones in the United States



US Dep of Energy: International Energy Conservation Code

The Cross-Covariance Function

Definition (Cross-Covariance Function)

Let X_t and Y_t be (weakly) stationary time series. The **cross-covariance function** (CCVF) of X_t and Y_t at lag k is given by

$$\gamma_{XY}(k) = \text{Cov}(X_t, Y_{t-k})$$

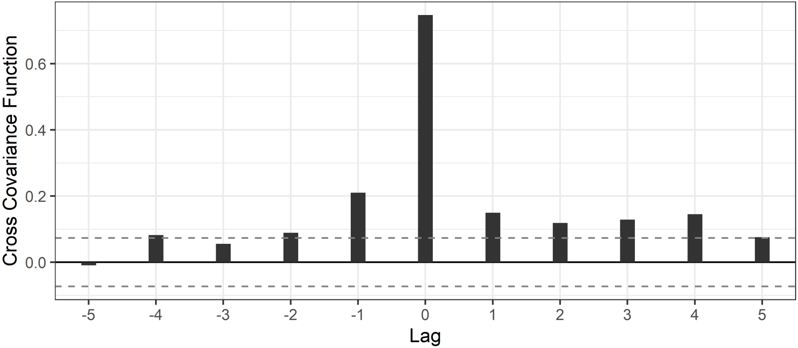
for $k \in \mathbb{Z}$

- Measures the linear dependence between X_t and Y_t at each lag k.

Estimated using

$$\hat{\gamma}_{XY}(k) = \frac{1}{n} \sum_{t=k+1}^{n} (X_t - \bar{X})(Y_{t-k} - \bar{Y})$$

CCF of the CLE and NYC maximum temperature series



How do we incorporate this information in the hypothesis test?

A Test for Two (Linearly) Dependent Series

Let X_t and Y_t be two univariate, zero mean stationary time series. We allow these series to be linearly dependent.

Define a new series $\mathrm{Z}_t = (X_t, Y_t)'$.

Denote the autocovariance function of Z_t as

$$\Gamma_{\mathrm{Z}}(k) = E[\mathrm{Z}_t \mathrm{Z}'_{t-k}] = \begin{bmatrix} \gamma_{Z_1 Z_1}(k) & \gamma_{Z_1 Z_2}(k) \\ \gamma_{Z_2 Z_1}(k) & \gamma_{Z_2 Z_2}(k) \end{bmatrix} = \begin{bmatrix} \gamma_{XX}(k) & \gamma_{XY}(k) \\ \gamma_{YX}(k) & \gamma_{YY}(k) \end{bmatrix}$$

Note that the Lund et al. [2009] test imposes independence of the two series, giving

$$\Gamma_Z(k) = \begin{bmatrix} \gamma_{XX}(k) & 0 \\ 0 & \gamma_{YY}(k) \end{bmatrix}$$

Define

$$\hat{\Lambda}_k = \text{vec}\ \Gamma_Z(k) = \begin{bmatrix} \hat{\gamma}_{XX}(k) \\ \hat{\gamma}_{YX}(k) \\ \hat{\gamma}_{XY}(k) \\ \hat{\gamma}_{YY}(k) \end{bmatrix}$$

for $0 \leq k \leq L$.

For $\hat{\Lambda}_0$, omit the $\hat{\gamma}_{XY}(0)$ to avoid later covariance matrix singularity.

Then, construct the $4L + 3$ dimensional vector $\hat{\Lambda}$ by stacking the $\hat{\Lambda}_k$ for $k \in \{0, 1, \dots, L\}$ in ascending order of k.

Ex: If I were interested in $L = 1$

$$\hat{\Lambda}_0 = \begin{bmatrix} \hat{\gamma}_{XX}(0) \\ \hat{\gamma}_{YX}(0) \\ \hat{\gamma}_{YY}(0) \end{bmatrix}, \quad \hat{\Lambda}_1 = \begin{bmatrix} \hat{\gamma}_{XX}(1) \\ \hat{\gamma}_{YX}(1) \\ \hat{\gamma}_{XY}(1) \\ \hat{\gamma}_{YY}(1) \end{bmatrix}, \quad \hat{\Lambda} = \begin{bmatrix} \hat{\gamma}_{XX}(0) \\ \hat{\gamma}_{YX}(0) \\ \hat{\gamma}_{YY}(0) \\ \hat{\gamma}_{XX}(1) \\ \hat{\gamma}_{YX}(1) \\ \hat{\gamma}_{XY}(1) \\ \hat{\gamma}_{YY}(1) \end{bmatrix}$$

Asymptotic Properties

Note that by Bartlett's Result, just as in the Lund et al. [2009] test

$$\sqrt{n}\left(\hat{\Lambda} - \Lambda\right) \xrightarrow{d} MVN\,(0, \mathrm{W})$$

where the entries of W are again computed using a multivariate analog of Bartlett's formula

$$\lim_{n\to\infty} n\mathrm{Cov}(\hat{\gamma}_{ab}(p), \hat{\gamma}_{cd}(q)) =$$

$$\sum_{r=-\infty}^{\infty} \gamma_{ac}(r)\gamma_{b,d}(r-p+q) + \gamma_{ad}(r+q)\gamma_{b,c}(r-p)$$

Construct a contrast matrix A such that

$$\hat{\Delta} = A'\hat{\Lambda} = \begin{bmatrix} \hat{\gamma}_{XX}(0) - \hat{\gamma}_{YY}(0) \\ \hat{\gamma}_{XX}(1) - \hat{\gamma}_{YY}(1) \\ \vdots \\ \hat{\gamma}_{XX}(L) - \hat{\gamma}_{YY}(L) \end{bmatrix}$$

Ex: If I were interested in $L = 1$

$$A' = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix}$$

$$\hat{\Delta} = \begin{bmatrix} 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} \hat{\gamma}_{XX}(0) \\ \hat{\gamma}_{YX}(0) \\ \hat{\gamma}_{YY}(0) \\ \hat{\gamma}_{XX}(1) \\ \hat{\gamma}_{YX}(1) \\ \hat{\gamma}_{XY}(1) \\ \hat{\gamma}_{YY}(1) \end{bmatrix} = \begin{bmatrix} \hat{\gamma}_{XX}(0) - \hat{\gamma}_{YY}(0) \\ \hat{\gamma}_{XX}(1) - \hat{\gamma}_{YY}(1) \end{bmatrix}$$

Asymptotics

We have that $\sqrt{n}\left(\hat{\Lambda} - \Lambda\right) \xrightarrow{d} MVN(0, \mathrm{W})$ as $n \to \infty$.

Usual multivariate results then give
$\sqrt{n}\hat{\Delta} = \sqrt{n}\mathrm{A}'\hat{\Lambda} \xrightarrow{d} MVN(0, \mathrm{A}'\mathrm{W}\mathrm{A})$ under the null hypothesis.

Thus

$$U_L^* = n\hat{\Delta}'(\mathrm{A}'\mathrm{W}\mathrm{A})^{-1}\hat{\Delta} \xrightarrow{d} \chi^2_{L+1}$$

under the null hypothesis of equality of autocovariances.

The Dependent Test

The previous findings give rise to a hypothesis test of equality of autocovariances that rejects the null hypothesis when $U_L^* = n\hat{\Delta}'(\mathrm{A}'\hat{\mathrm{W}}\mathrm{A})^{-1}\hat{\Delta}$ exceeds the α-th quantile of a χ^2_{L+1} distribution.

As before, $\hat{\mathrm{W}}$ is estimated using

$$\mathrm{Cov}(\hat{\gamma}_{ab}(p), \hat{\gamma}_{cd}(q)) \approx$$
$$\frac{1}{n} \sum_{r=-\lfloor n^{1/3} \rfloor}^{\lfloor n^{1/3} \rfloor} \hat{\gamma}_{ac}(r)\hat{\gamma}_{bd}(r-p+q) + \hat{\gamma}_{ad}(r+q)\hat{\gamma}_{bc}(r-p)$$

where the $\hat{\gamma}_{X_iX_j}(k)$ and $\hat{\gamma}_{Y_iY_j}(k)$ terms inside the sum are replaced with the null hypothesis estimate $\hat{\gamma}_{ij}(k) = \frac{1}{2}\left[\hat{\gamma}_{X_iX_j}(k) + \hat{\gamma}_{Y_iY_j}(k)\right]$

Why Does This Work?

Ex: If interested testing at $L = 1$

The asymptotic covariance matrix of $\hat{\Delta} = \begin{bmatrix} \hat{\gamma}_{XX}(0) - \hat{\gamma}_{YY}(0) \\ \hat{\gamma}_{XX}(1) - \hat{\gamma}_{YY}(1) \end{bmatrix}$ is given by

$$\begin{bmatrix} \begin{matrix} \text{Var}(\gamma_{XX}(0))+\text{Var}(\gamma_{YY}(0)) \\ -2\text{Cov}(\gamma_{XX}(0),\gamma_{YY}(0)) \end{matrix} & \begin{matrix} \text{Cov}(\gamma_{XX}(0),\gamma_{XX}(1))-\text{Cov}(\gamma_{YY}(0),\gamma_{XX}(1)) \\ -\text{Cov}(\gamma_{XX}(0),\gamma_{YY}(1))+\text{Cov}(\gamma_{YY}(0),\gamma_{YY}(1)) \end{matrix} \\ \begin{matrix} \text{Cov}(\gamma_{XX}(0),\gamma_{XX}(1))-\text{Cov}(\gamma_{YY}(0),\gamma_{XX}(1)) \\ -\text{Cov}(\gamma_{XX}(0),\gamma_{YY}(1))+\text{Cov}(\gamma_{YY}(0),\gamma_{YY}(1)) \end{matrix} & \begin{matrix} \text{Var}(\gamma_{XX}(1))+\text{Var}(\gamma_{YY}(1)) \\ -2\text{Cov}(\gamma_{XX}(1),\gamma_{YY}(1)) \end{matrix} \end{bmatrix}$$

Why Does This Work?

Note that the upper left term of the asymptotic covariance matrix contains $\text{Cov}(\gamma_{XX}(0), \gamma_{YY}(0))$ which can be estimated by

$$\text{Cov}(\hat{\gamma}_{XX}(0), \hat{\gamma}_{YY}(0)) \approx \frac{1}{n} \sum_{r=-\lfloor n^{1/3} \rfloor}^{\lfloor n^{1/3} \rfloor} 2\hat{\gamma}_{XY}^2(r)$$

So the covariance matrix contains information on the dependence structure between the two series!

The Independent vs Dependent Test Covariance Matrix

Ex: If interested testing at $L = 1$
When independent:

$$\begin{bmatrix} \text{Var}(\gamma_{XX}(0)) + \text{Var}(\gamma_{YY}(0)) & \text{Cov}(\gamma_{XX}(0), \gamma_{XX}(1)) + \text{Cov}(\gamma_{YY}(0), \gamma_{YY}(1)) \\ \text{Cov}(\gamma_{XX}(0), \gamma_{XX}(1)) + \text{Cov}(\gamma_{YY}(0), \gamma_{YY}(1)) & \text{Var}(\gamma_{XX}(1)) + \text{Var}(\gamma_{YY}(1)) \end{bmatrix}$$

but when dependent:

$$\begin{bmatrix} \begin{matrix} \text{Var}(\gamma_{XX}(0)) + \text{Var}(\gamma_{YY}(0)) \\ -2\text{Cov}(\gamma_{XX}(0), \gamma_{YY}(0)) \end{matrix} & \begin{matrix} \text{Cov}(\gamma_{XX}(0), \gamma_{XX}(1)) - \text{Cov}(\gamma_{YY}(0), \gamma_{XX}(1)) \\ -\text{Cov}(\gamma_{XX}(0), \gamma_{YY}(1)) + \text{Cov}(\gamma_{YY}(0), \gamma_{YY}(1)) \end{matrix} \\ \begin{matrix} \text{Cov}(\gamma_{XX}(0), \gamma_{XX}(1)) - \text{Cov}(\gamma_{YY}(0), \gamma_{XX}(1)) \\ -\text{Cov}(\gamma_{XX}(0), \gamma_{YY}(1)) + \text{Cov}(\gamma_{YY}(0), \gamma_{YY}(1)) \end{matrix} & \begin{matrix} \text{Var}(\gamma_{XX}(1)) + \text{Var}(\gamma_{YY}(1)) \\ -2\text{Cov}(\gamma_{XX}(1), \gamma_{YY}(1)) \end{matrix} \end{bmatrix}$$

Simulations

In the following simulations, we employ 1000 realizations of data generated from a VAR(1) process with parameters

$$\Phi = \begin{bmatrix} \phi_X & \rho_\phi \\ \rho_\phi & \phi_Y \end{bmatrix} \quad \text{and} \quad \Sigma = \begin{bmatrix} 1 & \rho_\Sigma \\ \rho_\Sigma & 1 \end{bmatrix}.$$

$$Z_t = \Phi Z_{t-1} + E_t \qquad E_t \overset{iid}{\sim} MVN(0, \Sigma)$$

where X_t and Y_t are taken as the first and second components of Z_t, respectively.

Size Studies

First, we take the two series as AR(1) processes of length $n = 1024$, with varying levels of dependence. Here $\phi_X = \phi_Y = 0.5$.

	Independent		Lag 0		Lag 1	
L	5	10	5	10	5	10
U_L	4.8	4.7	4.2	4.3	2.8	2.6
U_L^*	5.2	4.7	5.8	5.5	5.1	5.0

Any cross-correlation is given by $\rho_\Sigma = 0.2$ or $\rho_\phi = 0.2$.

Size Studies

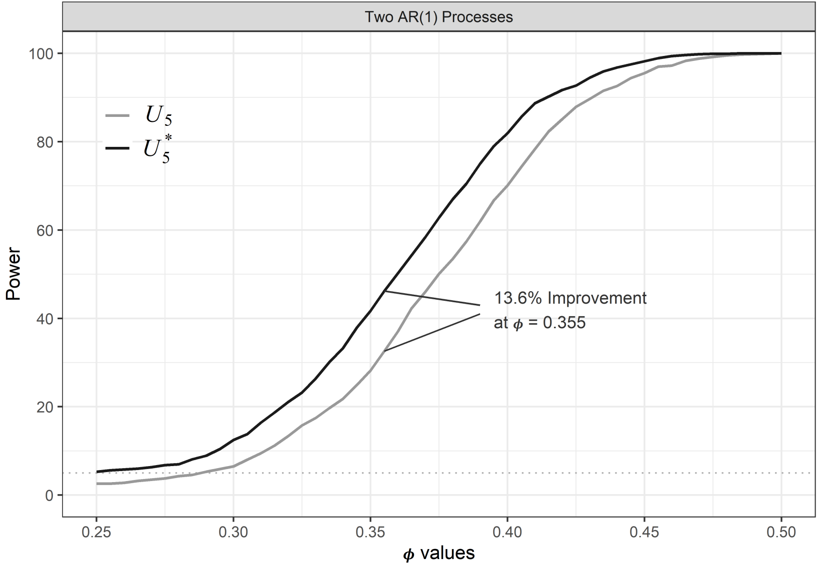
We again consider two AR(1) processes of length $n = 1024$. Here $\rho_\phi = 0.2$ and $\phi_X = \phi_Y = \phi$ across varying ϕ.

	ϕ									
	-0.5		-0.25		0		0.25		0.5	
	$n = 512$									
L	5		5		5		5		5	
U_L	2.0		2.7		2.3		2.3		2.2	
U_L^*	4.3		4.2		4.8		5.0		4.9	
	$n = 1024$									
L	5	10	5	10	5	10	5	10	5	10
U_L	2.6	2.1	2.5	2.4	2.7	2.4	2.6	2.2	2.1	1.9
U_L^*	5.1	4.8	5.6	4.5	5.4	4.3	5.3	4.5	5.1	4.6

Power Study

In the following power study X_t and Y_t are AR(1) processes with $\phi_X = 0.25$ and ϕ_Y is allowed to vary. Here $\rho_\phi = 0.2$ determines the level of dependence.

Power Study



Two AR(1) Processes
100
80
60
40
20
0
Power
U_5
U_5^*
13.6% Improvement
at ϕ = 0.355
0.25
0.30
0.35
0.40
0.45
0.50
ϕ values

The Tests Run on the CLE and NYC Series

Test	L	Statistic	df	p
Lund, et al - U_L	5	6.340	6	0.386
Proposed - U_L^*	5	15.312	6	0.018

Table: Tests for Equality of Autocovariances up to Lag 5

Additional Considerations

The same test has been developed for multivariate time series!

- Tests $H_0 : \gamma_{X_i X_j}(l) = \gamma_{Y_i Y_j}(l)$ vs $H_A : \gamma_{X_i X_j}(l) \neq \gamma_{Y_i Y_j}(l)$ for $1 \leq i, j \leq m$ and $1 \leq l \leq L$
- Theoretical results largely follow that presented today but with lots of careful *bookkeeping*!

Studied weighted variants of the test

- Different lags receive different emphasis
- Can help when series are close to non-stationary
- Similar to the ideas of Fisher and Gallagher [2012] or Hong [1996]

Additional Considerations

An order selection test for two dependent series is also proposed

$$\hat{S}_L^* = \max_{0 \le l \le L} \{\hat{U}_l^* - 2(l+1)\} \quad (1)$$

- Similar to results in Jin et al. [2019]
- The "$-2(l+1)$" acts a bit like the penalty term in AIC, a multivariate version exists too.
- The distribution of $\hat{S}_L^*$ follows a Chi-Square process but can be well approximated through bootstrapping
- The bootstrapping approach provides some level of robustness to the proposed methods for non-Gaussian data.

R Package

All of these methods are available in the R package `autocovarianceTesting`.

- Currently available on Github: `https://github.com/cirkovd/autocovarianceTesting/`
- Working its way to CRAN (hopefully soon).

Questions?

Email Dan at `cirkovd@tamu.edu` or
email Tom at `fishert4@miamioh.edu`

Slides available at `https://tjfisher19.github.io/`

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